

AN OBSTACLE PROBLEM FOR A CLASS OF MONGE-AMPÈRE TYPE FUNCTIONALS

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ABSTRACT. In this paper we study an obstacle problem for Monge-Ampère type functionals, whose Euler-Lagrange equations are a class of fourth order equations, including the affine maximal surface equations and Abreu's equation.

1. INTRODUCTION

Free boundary and obstacle problems for partial differential equations have been studied extensively in the past decades. For Monge-Ampère equations, obstacle problems were studied in [6, 13, 15] among others, and a related free boundary problem was studied in [5]. In this paper we consider an obstacle problem for the functional

$$(1.1) \quad J_\alpha(u) = \begin{cases} \int_\Omega [\det D^2 u]^\alpha - \alpha \int_\Omega f u, & \alpha > 0 \text{ and } \alpha \neq 1, \\ \int_\Omega \log \det D^2 u - \int_\Omega f u, & \alpha = 0, \end{cases}$$

where Ω is a bounded domain in \mathbb{R}^n and $f \in L^\infty(\Omega)$. For simplicity, we denote the nonlinear part of the functional (1.1) by $A_\alpha(u)$, see (2.4). We would like to study the maximization problem

$$(1.2) \quad J_\alpha(u) = \sup \{ J_\alpha(v) : v \in \mathcal{S}[\varphi, \psi] \},$$

where $\mathcal{S}[\varphi, \psi]$ is the class of functions

$$(1.3) \quad \mathcal{S}[\varphi, \psi] = \{ u \in C(\overline{\Omega}) : u \text{ convex}, u|_{\partial\Omega} = \varphi, Du(\Omega) \subset D\varphi(\overline{\Omega}), u \geq \psi \text{ in } \Omega \},$$

φ is a smooth, uniformly convex function defined on a neighborhood of $\overline{\Omega}$, ψ is an obstacle function, and $Du(\Omega)$ represents the image of the subgradients of u at all points $x \in \Omega$.

The Euler-Lagrange equations of (1.1) are a class of fourth order equations, that is,

$$(1.4) \quad U^{ij} w_{ij} = f,$$

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where (U^{ij}) is the cofactor matrix of the Hessian D^2u , and

$$(1.5) \quad w = [\det D^2u]^{-(1-\alpha)}, \quad \alpha \geq 0.$$

When $\alpha = \frac{1}{n+2}$, equation (1.4) is the affine mean curvature equation and the functional (2.4) is the affine area functional. When $\alpha = 0$, equation (1.4) is Abreu's equation arising from the study of Calabi's extremal metrics on toric Kähler manifolds [7, 8, 9, 10].

Due to their importance in geometry, variational problems of (1.1) have attracted much interest in recent years. In the case of $\alpha = \frac{1}{n+2}$, the variational problem without obstacle is the graph case of affine Plateau problem [20, 21], raised by Calabi and Chern. The case of $\alpha = 0$ has been treated in [22]. The obstacle problem of affine maximal surfaces was first introduced in [16]. In this paper, we obtain:

Theorem 1.1. *Suppose $n = 2$, $0 \leq \alpha \leq \frac{1}{n+2}$, and $f \in L^\infty(\Omega)$. Let φ be a smooth, uniformly convex function in Ω . If ψ is a convex function in Ω satisfying $\psi < \varphi$ on $\partial\Omega$, then there exists a unique maximizer of (1.2) which is strictly convex and $C^{1,\alpha}$ in Ω . Furthermore, if ψ is uniformly convex Ω , then the maximizer of (1.2) is $C^{1,1}$ in Ω .*

We remark that in higher dimensions, the problem is more complicated since Lemma 4.1 does not hold. Furthermore, in the case of $\alpha = 0$, the interior estimate in Lemma 2.3 remains open when $n > 2$. We will consider the higher dimensional cases and more general forms of the Monge-Ampère type functionals with $f = f(x, u, Du)$ in our forthcoming work.

This paper is organized as follows: In Section 2 we recall some preliminary results that will be used in subsequent sections. In addition, we show that how the functionals and equations change under a rotation in \mathbb{R}^{n+1} and obtain the a priori determinant estimates under the rotation transform, where the functionals have more general forms (2.15). In Section 3 we show that the maximizer of J_α can be approximated by a sequence of smooth maximizers of appropriate penalized functionals. In Section 4 we prove that the maximizer is strictly convex by an observation in [18, 21]. The proof of Theorem 1.1 is contained in Section 5, where the $C^{1,\alpha}$ and $C^{1,1}$ regularities are obtained, respectively.

2. PRELIMINARIES

2.1. Monge-Ampère measure. Let Ω be a bounded domain in \mathbb{R}^n and u be a convex function in Ω . The *normal mapping* of u , N_u , is a set-valued mapping defined as follows. For any point $x \in \Omega$, $N_u(x)$ is the set of slopes of supporting hyperplanes of u at x , that is,

$$(2.1) \quad N_u(x) = \{p \in \mathbb{R}^n : u(y) \geq u(x) + p \cdot (y - x), \quad \forall y \in \Omega\}.$$

For any Borel set $E \subset \Omega$, $N_u(E) = \bigcup_{x \in E} N_u(x)$. If u is C^1 , the normal mapping N_u is exactly the gradient mapping Du .

From the normal mapping we define the *Monge-Ampère measure* $\mu[u]$ by

$$(2.2) \quad \mu[u](E) = |N_u(E)|$$

for any Borel set $E \subset \Omega$, where the right hand side is the Lebesgue measure of $N_u(E)$. If u is C^2 smooth, we have $\mu[u] = (\det D^2u)dx$. In the non-smooth case, the Monge-Ampère measure $\mu[u]$ is a Radon measure, and is weakly continuous with respect to the convergence of convex functions, namely if a sequence of convex functions $\{u_i\}$ converges to a convex function u in L_{loc}^∞ , then for any closed $E \subset \Omega$,

$$(2.3) \quad \limsup_{i \rightarrow \infty} \mu[u_i](E) \leq \mu[u](E).$$

2.2. Existence and uniqueness of maximizer. Note that the functional J_α in (1.1) is well defined on the set of C^2 -smooth, convex functions. To study the maximization problem, we extend the functional J_α to the set $\mathcal{S}[\varphi, \psi]$ in (1.3), which is closed under the locally uniform convergence of convex functions. It is clear that the linear part in J_α is naturally defined. It suffices to extend the nonlinear part A_α to $\mathcal{S}[\varphi, \psi]$. If u is a convex function, u is almost everywhere twice-differentiable, i.e., the Hessian matrix (D^2u) exists almost everywhere. Denote the extended Hessian matrix by $\partial^2u(x) = D^2u(x)$ when u is twice differentiable at $x \in \Omega$ and $\partial^2u(x) = 0$ otherwise. As a Radon measure, $\mu[u]$ can be decomposed into a regular part and a singular part as follows,

$$\mu[u] = \mu_r[u] + \mu_s[u].$$

It was proved in [18] that the regular part $\mu_r[u]$ can be given explicitly by $\mu_r[u] = \det \partial^2u \, dx$ and hence $\det \partial^2u$ is a locally integrable function. Therefore for any $u \in \mathcal{S}[\varphi, \psi]$, we can define

$$(2.4) \quad A_\alpha(u) = \begin{cases} \int_\Omega [\det \partial^2u]^\alpha, & \alpha > 0, \\ \int_\Omega \log \det \partial^2u, & \alpha = 0. \end{cases}$$

Lemma 2.1. *Suppose $0 \leq \alpha \leq \frac{1}{n+2}$. J_α is upper semi-continuous, bounded and concave in $\mathcal{S}[\varphi, \psi]$. It follows that there exists a unique maximizer u_0 of (1.2).*

Proof. The proof for the cases $\alpha = \frac{1}{n+2}$ and $\alpha = 0$ can be found in [18, 23], respectively. One can check that the proof also holds for $0 < \alpha < \frac{1}{n+2}$. \square

2.3. Estimates for classical solutions. We include the following a priori estimates in [17, 18], which will be needed in subsequent sections, see also [8, 22] for the case of $\alpha = 0$. Consider the equation

$$(2.5) \quad \begin{aligned} U^{ij}w_{ij} &= f \quad \text{in } \Omega, \\ w &= [\det D^2u]^{\alpha-1}, \end{aligned}$$

where (U^{ij}) is the cofactor matrix of the Hessian matrix D^2u , and $\alpha \in [0, 1)$ is a constant.

Lemma 2.2. *Let $u \in C^4(\Omega) \cap C^{0,1}(\overline{\Omega})$ be a convex solution of (2.5) with $u = 0$ on $\partial\Omega$. Then for any $y \in \Omega$, we have the a priori estimate*

$$(2.6) \quad \det D^2 u(y) \leq C,$$

where C depends only on $n, \alpha, \text{dist}(y, \partial\Omega), \sup_{\Omega}(-u), \sup_{\Omega}|Du|$, and $\sup_{\Omega} f$.

Remark 2.1. In Lemma 2.2, the constant C is independent of $\inf_{\Omega} f$. Hence it is independent of f if $f \leq 0$. By Lemma 3.2, the maximizer u_0 of J_{α} can be locally approximated by smooth solutions of (2.5), and thus Lemma 2.2 still holds for non-smooth maximizers. When $\alpha = \frac{1}{n+2}$, the estimate (2.6) was previously proved in [18].

Remark 2.2. If $n = 2$, the assumption $u = 0$ on $\partial\Omega$ in Lemma 2.2 can be removed [18].

To prove that $\det D^2 u$ has a positive lower bound, we consider the Legendre transform u^* of u , which is a convex function defined in the domain $\Omega^* = N_u(\Omega)$, given by

$$(2.7) \quad u^*(y) = \sup\{x \cdot y - u(x) : x \in \Omega\}.$$

If u is strictly convex near $\partial\Omega$, u can be recovered from u^* by the same transform. If u is C^2 smooth at x , $y = Du(x)$ and $\det D^2 u(x) \neq 0$, then the Hessian matrix $D^2 u(x)$ is the inverse of the Hessian matrix $D^2 u^*(y)$, and

$$(2.8) \quad \det D^2 u(x) = [\det D^2 u^*(y)]^{-1}.$$

In particular, if u is a maximizer of the functional J_{α} , u^* is a maximizer of the dual functional

$$(2.9) \quad J_{\alpha}^*(u) = \begin{cases} \int_{\Omega^*} [\det D^2 u^*]^{1-\alpha} dy - \alpha \int_{\Omega^*} f(Du^*)(yDu^* - u^*) \det D^2 u^* dy, & \alpha > 0 \text{ and } \alpha \neq 1, \\ - \int_{\Omega^*} \det D^2 u^* \log \det D^2 u^* dy - \int_{\Omega^*} f(Du^*)(yDu^* - u^*) \det D^2 u^* dy, & \alpha = 0. \end{cases}$$

Therefore, if u^* is smooth, it satisfies the equation

$$(2.10) \quad U^{*ij} w_{ij}^* = \begin{cases} -\frac{\alpha}{1-\alpha} f(Du^*) \det D^2 u^*, & \alpha > 0 \text{ and } \alpha \neq 1, \\ -f(Du^*) \det D^2 u^*, & \alpha = 0, \end{cases}$$

where U^{*ij} is the cofactor matrix of $D^2 u^*$ and

$$(2.11) \quad w^* = \begin{cases} [\det D^2 u^*]^{-\alpha}, & \alpha > 0 \text{ and } \alpha \neq 1, \\ -\log \det D^2 u^*, & \alpha = 0. \end{cases}$$

By a similar argument to that of Lemma 2.2, we have the following result [17, 18, 22].

Lemma 2.3. *Let u^* be a smooth convex solution of (2.10) in Ω^* in dimension 2, $u^* = 0$ on $\partial\Omega^*$. Then for any $y \in \Omega^*$, we have the a priori estimate*

$$(2.12) \quad \det D^2 u^*(y) \leq C,$$

where C depends only on $\alpha, \text{dist}(y, \partial\Omega^*), \sup_{\Omega^*}|u^*|, \sup_{\Omega^*}|Du^*|$, and $\inf f$.

By (2.8) and (2.12), we have $\det D^2u \geq C$ has a positive lower bound. Note that the estimate depends on $\inf f$, but is independent of $\sup f$.

By Lemmas 2.2, 2.3 and the Caffarelli-Gutiérrez theory [4], we have the following Hölder and Sobolev space estimates.

Theorem 2.1. *Let $u \in C^4(\Omega)$ be a locally uniformly convex solution of (2.5).*

(i) *Assume $f \in L^\infty(\Omega)$. Then we have the estimate*

$$(2.13) \quad \|u\|_{W^{4,p}(\Omega')} \leq C,$$

for any $p > 1$ and $\Omega' \Subset \Omega$, where the constant C depends on $n, p, \sup_\Omega |f|, \text{dist}(\Omega', \partial\Omega)$, and the modulus of convexity of u .

(ii) *Assume $f \in C^\alpha(\Omega)$ for some $\alpha \in (0, 1)$. Then*

$$(2.14) \quad \|u\|_{C^{4,\alpha}(\Omega')} \leq C,$$

where C depends on $n, \alpha, \|f\|_{C^\alpha(\Omega)}, \text{dist}(\Omega', \partial\Omega)$, and the modulus of convexity of u .

Therefore, to prove the regularity of the maximizer u_0 in Lemma 2.1, it suffices to prove, in view of Lemmas 2.2, 2.3 and Theorem 2.1, that (a) the maximizer u_0 can be approximated by smooth solutions to equation (2.5) and (b) it is strictly convex. We will prove (a) and (b) in Sections 3 and 4, respectively.

2.4. Rotations in \mathbb{R}^{n+1} . In order to establish the estimate of the modulus of convexity, we need to treat convex functions as graphs in \mathbb{R}^{n+1} , and rotate the graphs in \mathbb{R}^{n+1} . When $\alpha = 1/(n+2)$, the affine maximal surface equation (1.4) is invariant under uni-modular transformations in \mathbb{R}^{n+1} . But this is not true for other α . It has been proved in [22] that for $\alpha = 0$, under the rotations in \mathbb{R}^{n+1} , equation (1.4) changes in a proper way such that the determinant estimate in Lemma 2.2 still holds.

For our purpose, we consider a more general functional

$$(2.15) \quad J_\alpha(u) = A_\alpha(u) - \int_\Omega F(x, u) dx,$$

where A_α is in (2.4), $F(x, t)$ is a function on $\Omega \times \mathbb{R}$. Let u be a locally critical point of the functional J_α , thus it satisfies (1.4) with the inhomogeneous term $f = F_t := \frac{\partial F}{\partial t}$.

Consider the rotation $Z = TX$, given by $z_1 = -x_{n+1}, z_{n+1} = x_1, z_i = x_i$ for $2 \leq i \leq n$. Assume the graph of u , $\mathcal{G}_u = \{(x, u(x)) : x \in \Omega\}$, can be represented by a convex function $z_{n+1} = v(z_1, \dots, z_n)$ in z -coordinates over a domain $\hat{\Omega}$. Following the computation in [23], v is a locally critical point of

$$(2.16) \quad \hat{J}_\alpha(v) = \hat{A}_\alpha(v) - \int_{\hat{\Omega}} F(v, z_2, \dots, z_n, -z_1),$$

where

$$\hat{A}_\alpha(v) = \begin{cases} \int_{\hat{\Omega}} [\det D^2 v]^\alpha |v_1|^{1-(n+2)\alpha} dz, & \alpha > 0, \\ \int_{\hat{\Omega}} [\log \det D^2 v - \frac{n+2}{2} \log(v_1^2)] (v_1^2)^{\frac{1}{2}} dz, & \alpha = 0. \end{cases}$$

When $\alpha > 0$, by computing the Euler equation, we can obtain the corresponding equation for v , that is,

$$(2.17) \quad \begin{aligned} \alpha v_1^{1-\alpha(n+2)} V^{ij} (d^{\alpha-1})_{ij} + (1-\alpha)\alpha(n+2)(1-\alpha(n+2)) v_1^{-\alpha(n+2)-1} v_{11} d^\alpha \\ + (1-\alpha(n+2))(2\alpha-2) v_1^{-\alpha(n+2)} (d^\alpha)_1 = F_t, \end{aligned}$$

or equivalently, denoting $\lambda = 1 - \alpha(n+2)$,

$$(2.18) \quad V^{ij} (d^{\alpha-1})_{ij} = g + \mathcal{F}_t,$$

where (V^{ij}) is the cofactor matrix of (v_{ij}) , $d = \det D^2 v$ and

$$\begin{aligned} g &= 2\lambda(1-\alpha) d^\alpha v^{ij} v_{ij1} \frac{1}{v_1} - (1-\alpha)(n+2) \lambda d^\alpha \frac{v_{11}}{v_1^2}, \\ \mathcal{F}_t &= \alpha^{-1} \frac{F_t}{v_1^\lambda}, \quad F_t = \frac{\partial F}{\partial t}(v, z_2, \dots, z_n, -z_1). \end{aligned}$$

When $\alpha = 0$, by a similar computation we obtain (2.18) with $\mathcal{F}_t = F_t/v_1$.

2.5. A priori estimates. In this subsection, we obtain the a priori determinant estimates under the rotation transform $Z = TX$. Let v be a smooth solution of (2.18) satisfying

$$(2.19) \quad v \geq 0, \quad v \geq z_1, \quad v_1 \geq 0,$$

and $v(0)$ is as small as we want such that for the positive constant s and h in $(0, 1/2)$, $\hat{\Omega}_{s,h}$ is a nonempty open set, where

$$(2.20) \quad \hat{\Omega}_{s,h} = \{z : v(z) < sz_1 + h\}.$$

Set $\hat{v} := v - sz_1 - h$, then $\hat{\Omega}_{s,h} = \{z : \hat{v}(z) < 0\}$ and \hat{v} satisfies

$$(2.21) \quad \hat{V}^{ij} (\hat{d}^{\alpha-1})_{ij} = \hat{g} + \hat{\mathcal{F}}_t,$$

where (\hat{V}^{ij}) is the cofactor matrix of (\hat{v}_{ij}) , $\hat{d} = \det D^2 \hat{v}$ and

$$\begin{aligned} \hat{g} &= 2\lambda(1-\alpha) \hat{d}^\alpha \frac{\hat{v}^{ij} \hat{v}_{ij1}}{\hat{v}_1 + s} - (1-\alpha)(n+2) \lambda \hat{d}^\alpha \frac{\hat{v}_{11}}{(\hat{v}_1 + s)^2}, \\ \hat{\mathcal{F}}_t &= \alpha^{-1} \frac{\hat{F}_t}{(\hat{v}_1 + s)^\lambda}, \quad \hat{F}_t = \frac{\partial F}{\partial t}(\hat{v} + sz_1 + h, z_2, \dots, z_n, -z_1). \end{aligned}$$

Lemma 2.4. Assume $0 \leq \alpha \leq \frac{1}{n+2}$. Let \hat{v} be a smooth solution of (2.21) in $\hat{\Omega}_{s,h}$ and $\hat{v} = 0$ on $\partial \hat{\Omega}_{s,h}$. Then for any $z \in \hat{\Omega}_{s,h}$, we have the a priori estimate

$$(2.22) \quad \det D^2 \hat{v} \leq C,$$

where C depends only $n, \alpha, \text{dist}(z, \partial \hat{\Omega}_{s,h}), \sup_{\hat{\Omega}_{s,h}} |\hat{v}|, \sup_{\hat{\Omega}_{s,h}} |D\hat{v}|$ and $\sup \hat{F}_t$.

Proof. When $\alpha = \frac{1}{n+2}$, the estimate (2.22) easily follows from the affine invariant property. Note that in this case, $\lambda = 0$ and \hat{g} in (2.21) vanishes. The case of $\alpha = 0$ was contained in [22]. Here we give a proof for the remaining case $0 < \alpha < \frac{1}{n+2}$ as follows. Let

$$(2.23) \quad \eta = \log w - \beta \log(-\hat{v}) - A|D\hat{v}|^2,$$

where $w = \hat{d}^{\alpha-1}$, and β, A are positive constants to be determined later. Since $\eta \rightarrow +\infty$ on $\partial\hat{\Omega}_{s,h}$, it attains a minimum at some point $z_0 \in \hat{\Omega}_{s,h}$. At z_0 , we then have

$$(2.24) \quad 0 = \eta_i = \frac{w_i}{w} - \frac{\eta \hat{v}_i}{\hat{v}} - 2A\hat{v}_k\hat{v}_{ki},$$

$$(2.25) \quad 0 \leq [\eta_{ij}] = \left[\frac{w_{ij}}{w} - \frac{w_i w_j}{w^2} - \frac{\beta \hat{v}_{ij}}{\hat{v}} + \frac{\beta \hat{v}_i \hat{v}_j}{\hat{v}^2} - 2A\hat{v}_{ki}\hat{v}_{kj} - 2A\hat{v}_k\hat{v}_{kij} \right]$$

as a matrix. Since $w = [\det D^2\hat{v}]^{\alpha-1}$, we have

$$(2.26) \quad \hat{v}^{ij}\hat{v}_{kij} = (\log \det D^2\hat{v})_k = \frac{1}{\alpha-1} \frac{w_k}{w},$$

where $(\hat{v}^{ij}) = \hat{d}^{-1}(V^{ij})$ is the inverse of $D^2\hat{v}$. We may assume that $\hat{d} > 1$, otherwise the proof is done. Hence,

$$(2.27) \quad \frac{\hat{v}^{ij}w_{ij}}{w} = \frac{\hat{g} + \hat{\mathcal{F}}_t}{\hat{d}^\alpha} \leq -2\lambda \frac{w_1}{w} (\hat{v}_1 + s)^{-1} - (1-\alpha)(n+2)\lambda \frac{\hat{v}_{11}}{(\hat{v}_1 + s)^2} + \frac{\sup \hat{F}_t}{\alpha(\hat{v}_1 + s)^\lambda}.$$

Therefore, we obtain

$$(2.28) \quad \begin{aligned} 0 &\leq \hat{v}^{ij}\eta_{ij} \\ &\leq \frac{\sup \hat{F}_t}{\alpha(\hat{v}_1 + s)^\lambda} + \frac{\lambda(\alpha-1)(n+2)\hat{v}_{11}}{(\hat{v}_1 + s)^2} - 4A\lambda \sum_{k=1}^n \frac{\hat{v}_{1k}\hat{v}_k}{\hat{v}_1 + s} - 2\lambda\beta \frac{\hat{v}_1}{(\hat{v}_1 + s)\hat{v}} \\ &\quad - \frac{\beta n}{\hat{v}} - \left(2A \triangle \hat{v} - \frac{4A^2\alpha}{1-\alpha} \hat{v}_{ij}\hat{v}_i\hat{v}_j \right) - \left(4A\beta - \frac{2A\beta}{1-\alpha} \right) \frac{|D\hat{v}|^2}{\hat{v}} - (\beta^2 - \beta) \frac{\hat{v}^{ij}\hat{v}_i\hat{v}_j}{\hat{v}^2} \\ &\leq \frac{\sup \hat{F}_t}{\alpha(\hat{v}_1 + s)^\lambda} - 2\lambda\beta \frac{\hat{v}_1}{(\hat{v}_1 + s)\hat{v}} - \frac{\beta n}{\hat{v}} - \frac{A}{2} \triangle \hat{v} - \left(4A\beta - \frac{2A\beta}{1-\alpha} \right) \frac{|D\hat{v}|^2}{\hat{v}}, \end{aligned}$$

with the choice of $\beta > 1$ and A small enough such that

$$(2.29) \quad \frac{A}{2} \triangle \hat{v} \geq \frac{4A^2\alpha}{1-\alpha} \hat{v}_{ij}\hat{v}_i\hat{v}_j + CA^2\hat{v}_{11},$$

where C is a constant depending only on n, α and $|D\hat{v}|$. Observing that

$$(2.30) \quad \frac{\hat{v}_1}{(\hat{v}_1 + s)\hat{v}} = \frac{1}{\hat{v}} - \frac{s}{(\hat{v}_1 + s)\hat{v}},$$

by choosing β large enough such that

$$(2.31) \quad (-\hat{v})(\hat{v}_1 + s)^{1-\lambda} \sup \hat{F}_t \leq 2s\alpha\lambda\beta,$$

we have

$$(2.32) \quad -\frac{\beta(n+2\lambda)}{\hat{v}} - \frac{A}{2} \triangle \hat{v} - \left(4A\beta - \frac{2A\beta}{1-\alpha} \right) \frac{|D\hat{v}|^2}{\hat{v}} \geq 0,$$

which implies

$$(2.33) \quad (-\hat{v}) \triangle \hat{v} \leq C.$$

It follows that $\eta(z) \geq \eta(z_0) \geq -C$ and so (2.22) holds. \square

3. APPROXIMATIONS

Let u_0 be the maximizer of (1.2). In this section, we prove that u_0 can be approximated by a sequence of smooth solutions to equation (2.5). The approximation enables us to apply the a priori estimates in Section 2. For Monge-Ampère equations, or general second order equations, one can obtain the approximation from a perturbation of the equation. However, the perturbation does not work for fourth order equations because of the lack of maximum principle. We will construct the approximation using a penalty method to the functionals. We also need to deal with the difficulty coming from the obstacle.

3.1. Obstacle approximation. Let u_0 be the maximizer of J_α in $\mathcal{S}[\varphi, \psi]$. We construct a sequence of penalized functionals whose maximizers do not contact the obstacle and approximate u_0 . Let $\mathcal{S}[\varphi, u_0]$ be the set of convex functions with u_0 as the obstacle, namely,

$$(3.1) \quad \mathcal{S}[\varphi, u_0] = \{v \in C(\overline{\Omega}) : v \text{ convex}, v|_{\partial\Omega} = \varphi, Dv(\Omega) \subset D\varphi(\overline{\Omega}), v \geq u_0 \text{ in } \Omega\},$$

where φ is a smooth, uniformly convex function defined on a neighborhood of $\overline{\Omega}$.

Lemma 3.1. *Suppose $0 \leq \alpha \leq \frac{1}{n+2}$. There exists a sequence of functions $\{u_i\}$ in $\mathcal{S}[\varphi, u_0]$ such that each u_i is the maximizer of the functional*

$$J_\alpha^i(v) = J_\alpha(v) - \int_{\Omega} G_i(x, v), \quad v \in \mathcal{S}[\varphi, u_0]$$

and $u_i \rightarrow u_0$ as $i \rightarrow \infty$, where $G_i(x, t)$ is a smooth, convex function monotone decreasing in t . Furthermore, there is no obstacle for u_i in Ω , i.e., $u_i(x) > u_0(x)$, $x \in \Omega$.

Proof. First, we consider a penalized problem. The idea is inspired by [16]. Define

$$(3.2) \quad J_{\alpha,g}(v) = J_\alpha(v) - \int_{\Omega} G(x, v),$$

where $G(x, t)$ is a smooth, convex function monotone decreasing in t such that

$$(3.3) \quad G(x, t) \geq a(x)(t - u_0(x))^{-n} \quad \text{for } t > u_0(x), \quad x \in \Omega.$$

Here a is a positive function in Ω , with $a(x) \rightarrow 0$ fast enough as $x \rightarrow \partial\Omega$ such that the set $\{v \in \mathcal{S}[\varphi, u_0] : J_{\alpha,g}(v) > -\infty\} \neq \emptyset$. It is clear that $J_{\alpha,g}$ is still concave, upper semi-continuous and bounded from above. Hence there is a unique maximizer v_g to the problem

$$(3.4) \quad \sup\{J_{\alpha,g}(v) : v \in \mathcal{S}[\varphi, u_0]\}.$$

We claim that for any $x \in \Omega$,

$$(3.5) \quad v_g(x) > u_0(x).$$

Indeed, if there is a point $x_0 \in \Omega$ such that $v_g(x_0) = u_0(x_0)$, by convexity the graphs of v_g and u_0 are bounded by the cone \mathcal{K} and the hyperplane \mathcal{P} , where \mathcal{K} has the vertex at $(x_0, u_0(x_0))$ and passes through $(\partial\Omega, u_0|_{\partial\Omega})$, and \mathcal{P} is the support plane of u_0 at x_0 . Then we have $|v_g(x) - u_0(x)| \leq C|x - x_0|$. Hence by the assumption on $G(x, t)$,

$$(3.6) \quad \int_{\Omega} G(x, v_g(x)) \geq C \int_{\Omega} |x - x_0|^{-n} = \infty.$$

That is, v_g cannot be a maximizer.

Replacing G by $\varepsilon_i G$ for a sequence $\varepsilon_i \rightarrow 0$, accordingly there exists a sequence of maximizers v_{ε_i} to (3.4). Since u_0 is itself a maximizer, we have $v_{\varepsilon_i} \rightarrow u_0$ as $\varepsilon_i \rightarrow 0$ by the concavity of the functional J_{α} . Hence, the sequence u_i can be chosen from v_{ε_i} . \square

Remark 3.1. If u_i is smooth, it satisfies the equation

$$(3.7) \quad L[u] = f + g_i \quad \text{in } \Omega,$$

where L is the operator in (1.4), and $g_i = \frac{\partial}{\partial t} G_i(x, t)$ at $t = u_i(x)$. In the later proof of strict convexity, we will need the upper bound estimate for the determinant of $D^2 u_0$ which depends on $\sup f$. Since $g_i < 0$ in the above approximation, the estimate in Section 2 still applies when turning to the sequence u_i .

When studying the strict convexity of enclosed convex hypersurfaces with maximal affine area, one can assume u_0 is equal to a linear function ℓ on $\partial\Omega$ [16], then the above proof can be simplified.

Remark 3.2. In fact, the approximation in Lemma 3.1 applies on any subdomain $\Omega' \subset \Omega$. Instead of considering the boundary φ , one can consider

$$\mathcal{S}_{\Omega'}[u_0] = \{v \in C(\overline{\Omega}') : v \text{ convex}, v|_{\partial\Omega'} = u_0|_{\partial\Omega'}, Dv(\Omega') \subset Du_0(\overline{\Omega}'), v \geq u_0\},$$

and then obtain a local approximation sequence.

3.2. Smooth approximation. Let u be the maximizer of (3.4). From the obstacle approximation, u is also the maximizer of (3.2) over the set

$$(3.8) \quad \mathcal{S}[\varphi, \Omega] = \{v \in C(\overline{\Omega}) : v \text{ convex}, v|_{\partial\Omega} = \varphi|_{\partial\Omega}, Dv(\Omega) \subset D\varphi(\overline{\Omega})\}.$$

In this subsection, we prove that u can be approximated by smooth solutions of

$$(3.9) \quad U^{ij} w_{ij} = f(x, u),$$

where U^{ij} is the cofactor of $D^2 u$ and $w = [\det D^2 u]^{\alpha-1}$. This approximation enables us to apply the a priori estimates in Section 2.

Lemma 3.2. *Let u be the maximizer of (3.4). Suppose $\partial\Omega$ is Lipschitz continuous. Then there exists a sequence of smooth solutions to equation (3.9) converging locally uniformly to the maximizer u .*

To prove the approximation, first we recall the existence and regularity of solutions of the following second boundary value problem [18]. Let $B = B_R(0)$ be a ball such that $\Omega \Subset B_{R-1}(0)$ and ϕ is a smooth, uniformly convex function in B and $\phi = c^*$ is constant on ∂B . Let

$$(3.10) \quad H(t) = (1 - t^2)^{-2n}$$

be a nonnegative smooth function in the interval $(-1, 1)$. When $|t| > 1$, we can formally define $H(t) = +\infty$. Extend the function f in (3.9) to B such that

$$(3.11) \quad f = \begin{cases} f(x) & x \in \Omega, \\ H'(u - \phi(x)) & x \in B \setminus \Omega. \end{cases}$$

Lemma 3.3. *Suppose $\partial\Omega$ is Lipschitz continuous. Then there is a uniformly convex solution $u \in W_{loc}^{4,p}(B) \cap C^{0,1}(\overline{B})$ (for all $p < \infty$) with $\det D^2u \in C^0(\overline{\Omega})$ of the boundary value problem*

$$(3.12) \quad \begin{aligned} U^{ij}w_{ij} &= f(x, u) \quad \text{in } B, \\ u &= \phi (= c^*) \quad \text{on } \partial B, \\ w &= 1 \quad \text{on } \partial B. \end{aligned}$$

The existence and regularity of solutions of (3.12) was previously obtained in [18, 21] for $\alpha = \frac{1}{n+2}$, and [22] for $\alpha = 0$. The crucial ingredient is to establish

$$(3.13) \quad |f(x, u)| \leq C$$

for some constant $C > 0$ independent of u . Once f is bounded, the regularity and existence of solutions follow easily from [18]. The global $C^{4,\alpha}$ regularity was recently proved in [20]. Following the argument in [18], one can easily check the proof works for all $\alpha \in (0, \frac{1}{n+2})$.

Now, we show that the maximizer of $J_\alpha(u)$ can be approximated by smooth solutions to equation (3.9).

Proof of Lemma 3.2. By assumption φ is smooth, uniformly convex in a neighborhood of Ω , so we can extend it to $B = B_R$ such that φ is convex in B , $\varphi \in C^{0,1}(\overline{B})$ and φ is constant on ∂B . Replacing φ by $\varphi + (|x| - R + \frac{1}{2})_+^2$, where

$$\left(|x| - R + \frac{1}{2}\right)_+ = \max \left\{ |x| - R + \frac{1}{2}, 0 \right\},$$

we also assume that φ is uniformly convex in $\{x \in \mathbb{R}^n : R - \frac{1}{2} < |x| < R\}$. Consider the second boundary value problem (3.12) with

$$(3.14) \quad f_j(x, u) = \begin{cases} f & \text{in } \Omega \\ H'_j(u - \varphi) & \text{in } B \setminus \Omega, \end{cases}$$

where $H_j(t) = H(4^j t)$ and H is defined by (3.10). By Lemma 3.3 there is a solution u_j satisfying

$$(3.15) \quad |u_j - \varphi| \leq 4^{-j}, \quad x \in B \setminus \Omega.$$

By the convexity, u_j sub-converges to a convex function \bar{u} in B as $j \rightarrow \infty$. Note that $\bar{u} = \varphi$ in $B \setminus \Omega$. Hence, $\bar{u} \in \mathcal{S}[\varphi, \Omega]$ when restricted in Ω . Using a similar argument as in [21] and [22], one can show that \bar{u} is the maximizer of (3.2) over the set (3.8). By the uniqueness of maximizer, we obtain $\bar{u} = u$. The main ingredients of the argument in [21] are the upper semicontinuity and the concavity of the functional (3.2), which hold for all $\alpha \in [0, \frac{1}{n+2}]$, see Lemma 2.1. \square

4. STRICT CONVEXITY

In this section, we prove the strict convexity of u_0 in dimension two. Let \mathcal{G}_0 be the graph of u_0 . If u_0 is not strictly convex, then \mathcal{G}_0 contains a line segment. Let $\ell(x)$ be a tangent function of u_0 at the segment and denote by

$$(4.1) \quad \mathcal{C} = \{x \in \Omega : u_0(x) = \ell(x)\}$$

the contact set. The set $\mathcal{C} \subset \mathbb{R}^2$ is bounded and convex.

We say a point $x_0 \in \partial U$ is an extreme point of a bounded convex domain $U \subset \mathbb{R}^n$ if there is a hyperplane P such that $\{x_0\} = P \cap \partial U$, namely the intersection $P \cap \partial U$ is the single point x_0 . We divide our discussion into the following two cases:

- (a) : \mathcal{C} has an extreme point x_0 , which is an interior point of Ω ;
- (b) : All extreme points of \mathcal{C} lie on $\partial\Omega$.

We will rule out the possibility of both cases, and thus u_0 is strictly convex. The basic observation is that a convex function with a bounded Monge-Ampère measure is differentiable at any point on its graph, not lying on a line segment joining two boundary points, [1]. In dimension two, recall the following

Lemma 4.1 ([18]). *Suppose u is a nonnegative convex function in a domain $\Omega \subset \mathbb{R}^2$. The origin $0 \in \Omega$ is an interior point. u satisfies $u > 0$ on $\partial\Omega$, $u(0) = 0$ and $u(x_1, 0) \geq |x_1|$. Then the Monge-Ampère measure $\mu[u]$ cannot be a bounded function.*

4.1. Strict convexity I. First we rule out the possibility that \mathcal{G}_0 contains a line segment with one endpoint in the interior of Ω .

Lemma 4.2. *\mathcal{C} contains no extreme points in the interior of Ω .*

Proof. The proof is by contradiction arguments as in [17, 22]. Without loss of generality, we may assume that $\ell(x) = 0$, the origin is an extreme point of \mathcal{C} and the segment $\{(x_1, 0) : 0 \leq x_1 \leq 1\} \subset \mathcal{C}$. From the approximation argument, we can choose a sequence of functions $\{u_k\}$

converging to u_0 such that u_k is a solution of (3.7). Let \mathcal{G}_k be the graph of u_k . Then \mathcal{G}_k converges in the Hausdorff distance to \mathcal{G}_0 .

For $\varepsilon > 0$ small enough, let

$$(4.2) \quad \ell_\varepsilon = -\varepsilon x_1 + \varepsilon, \quad \text{and } \Omega_\varepsilon = \{u < \ell_\varepsilon\}.$$

Let T_ε be a coordinates transformation that normalizes the domain Ω_ε . Define

$$(4.3) \quad u_\varepsilon(x) = \frac{1}{\varepsilon} u(T_\varepsilon^{-1}(x)), \quad u_{k,\varepsilon} = \frac{1}{\varepsilon} u_k(T_\varepsilon^{-1}(x)), \quad x \in \tilde{\Omega}_\varepsilon,$$

where $\tilde{\Omega}_\varepsilon = T_\varepsilon(\Omega_\varepsilon)$ is normalized. After this transformation we have the following observations:

(i) The equation $U^{ij}w_{ij} = f$ with $w = [\det D^2u]^{\alpha-1}$, $0 \leq \alpha \leq \frac{1}{4}$, will become

$$(4.4) \quad \tilde{U}^{ij}\tilde{w}_{ij} = \tilde{f},$$

where \tilde{U}^{ij} is the cofactor of $D^2\tilde{u}$,

$$\tilde{w} = [\det D^2\tilde{u}]^{\alpha-1}, \quad \text{and } \tilde{f} = |T_\varepsilon|^{-2\alpha} \varepsilon^{1-2\alpha} f.$$

In fact, since T_ε normalizes Ω_ε , $|T_\varepsilon|^{-1} \leq |\Omega_\varepsilon| \leq C$. Therefore, $\tilde{f} \rightarrow 0$ as $\varepsilon \rightarrow 0$.

(ii) Denote by \mathcal{G}_ε and $\mathcal{G}_{k,\varepsilon}$ the graphs of u_ε and $u_{k,\varepsilon}$, respectively. Taking $k \rightarrow \infty$, it is clear that $u_{k,\varepsilon} \rightarrow u_\varepsilon$ and $\mathcal{G}_{k,\varepsilon}$ converges in the Hausdorff distance to \mathcal{G}_ε . Then taking $\varepsilon \rightarrow 0$, we have that the domain $\tilde{\Omega}_\varepsilon$ sub-converges to a normalized domain $\tilde{\Omega}$ and u_ε sub-converges to a convex function \tilde{u} defined in $\tilde{\Omega}$. We also have \mathcal{G}_ε sub-converges in the Hausdorff distance to a convex surface $\tilde{\mathcal{G}}_0 \in \mathbb{R}^3$.

(iii) By a rotation of coordinates, the convex surface $\tilde{\mathcal{G}}_0$ satisfies

$$(4.5) \quad \tilde{\mathcal{G}}_0 \subset \{y_1 \geq 0\} \cap \{y_3 \geq 0\}$$

and $\tilde{\mathcal{G}}_0$ contains two segments

$$(4.6) \quad \{(0, 0, y_3) : 0 \leq y_3 \leq 3\}, \quad \{(y_1, 0, 0) : 0 \leq y_1 \leq 1\}.$$

Hence, by (i)–(iii) we can assume that there is a sequence of solutions \tilde{u}_k of

$$(4.7) \quad U^{ij}w_{ij} = \varepsilon_k f \quad \text{in } \tilde{\Omega}_k,$$

where $w = [\det D^2u]^{\alpha-1}$, and $\varepsilon_k \rightarrow 0$ such that the normalized domain $\tilde{\Omega}_k$ converges to $\tilde{\Omega}$, \tilde{u}_k converges to \tilde{u} and the graph of \tilde{u}_k , denoted by $\tilde{\mathcal{G}}_k$ converges in the Hausdorff distance to $\tilde{\mathcal{G}}_0$.

Note that in y -coordinates, $\tilde{\mathcal{G}}_0$ is not a graph of a function near the origin. By adding some linear function to \tilde{u}_k and \tilde{u} and making a rotation of coordinates in \mathbb{R}^3 , i.e., $z_i = R_{ij}y_j$, where (R_{ij}) is a 3×3 rotation matrix, $\tilde{\mathcal{G}}_k, \tilde{\mathcal{G}}_0$ can be represented by $z_3 = v_k(z_1, z_2), z_3 = v(z_1, z_2)$, respectively [22]. Moreover, v_k is a solution of the equation given in §2.4 near the origin, v satisfies

$$(4.8) \quad v \geq \frac{1}{2}|z_1|, \quad \text{and} \quad v(z_1, 0) = \frac{1}{2}|z_1|.$$

As we know that $\tilde{\mathcal{G}}_k$ converges in the Hausdorff distance to $\tilde{\mathcal{G}}_0$, in the new coordinates, v_k converges locally uniformly to v . Let $\tilde{\mathcal{C}} = \{(z_1, z_2), v(z_1, z_2) = 0\}$, and

$$(4.9) \quad L = \{(z_1, z_2, 0) : (z_1, z_2) \in \tilde{\mathcal{C}}\}$$

in z -coordinates. L could be a single point (Case I) or a segment on z_2 -axis (Case II).

Case I: In this case, v is strictly convex at $(0, 0)$. The strict convexity implies that Dv is bounded on the sub-level set $S_{h,v}(0)$ for small $h > 0$. Hence, by locally uniform convergence, Dv_k are uniformly bounded on $S_{h/2,v_k}(0)$. By Lemma 2.4, we have the determinant estimate

$$(4.10) \quad \det D^2 v_k \leq C$$

near the origin, where the constant C is uniform with respect to k . By the weak continuity of Monge-Ampère measure, $\mu[v] \leq C$ near the origin. The contradiction follows by Lemma 4.1.

Case II: In this case, L is a segment, we may also assume that 0 is an end point of L , i.e.,

$$\tilde{\mathcal{C}} = \{(0, z_2) : -1 \leq z_2 \leq 0\}.$$

Define the linear function

$$(4.11) \quad \ell_\varepsilon(z) = \delta_\varepsilon z_2 + \varepsilon$$

and $\omega_\varepsilon = \{z : v(z) \leq \ell_\varepsilon\}$, where $\delta_\varepsilon, \varepsilon$ are chosen such that $\varepsilon \delta_\varepsilon^{-1} \rightarrow 0$ as $\varepsilon \rightarrow 0$. By taking the similar transformations and normalizations as in (4.2), (4.3) with respect to z_2 direction, one can reduce Case II to Case I. The proof is then finished. □

4.2. Strict convexity II. Next, we rule out the possibility of case (b) that all extreme points of \mathcal{C} lie on the boundary $\partial\Omega$. Recall the definition of \mathcal{C} in (4.1), and define the set $T := \{x \in \Omega : u(x) = \psi(x)\}$, where ψ is the obstacle.

Lemma 4.3. *Let $u_0 \in \mathcal{S}[\varphi, \psi]$ be the maximizer. The obstacle ψ is a convex function in Ω satisfying $\psi < \varphi$ on $\partial\Omega$. If all extreme points of \mathcal{C} lie on the boundary $\partial\Omega$, then $\text{dist}(\overline{\mathcal{C}}, \overline{T}) > c_0$ for some positive constant c_0 .*

Proof. This follows easily from the convexity. □

Lemma 4.4. *Assume that φ is uniformly convex in a neighborhood of Ω . then \mathcal{G}_0 contains no line segments with both endpoints on $\partial\mathcal{G}_0$.*

Proof. By Lemma 4.3, we can restrict our discussion on a sub-domain $\Omega' \subset \Omega$ satisfying $\text{dist}(\Omega', T) > c_0$ and $\{\text{extreme points of } \mathcal{C}\} \subset \partial\Omega' \cap \partial\Omega$. Let u_0 be the maximizer of J_α and

$$(4.12) \quad \bar{S}[u_0, \Omega'] := \{v \in C(\overline{\Omega'}) : v \text{ convex}, v_{\partial\Omega'} = u_0, N_v(\Omega') \subset N_{u_0}(\overline{\Omega'})\}.$$

Note that since $\text{dist}(\Omega', T) > c_0$, when restricting on Ω' , u_0 is naturally a maximizer of J_α over $\bar{S}[u_0, \Omega']$ without obstacle. Therefore, we can apply a similar local approximation in [21] as follows:

Claim: There exists a sequence of smooth, uniformly convex solutions $u_m \in W^{4,p}(\Omega')$ ($\forall p < \infty$) of

$$(4.13) \quad U^{ij} w_{ij} = f + \beta_m \chi_{D_m} \quad \text{in } \Omega'$$

such that

$$(4.14) \quad |u_m - u| \rightarrow 0 \quad \text{uniformly in } \Omega',$$

where $D_m = \{x \in \Omega' : \text{dist}(x, \partial\Omega') < 2^{-m}\}$, χ is the characteristic function, and $\beta_m > 0$ is a constant. Furthermore, we can choose β_m sufficiently large ($\beta_m \rightarrow \infty$ as $m \rightarrow \infty$) such that for any compact, proper subset $K \subset N_{u_0}(\Omega')$,

$$(4.15) \quad K \subset N_{u_m}(\Omega')$$

provided m is sufficiently large, where N_u is the normal mapping introduced in Section 2.

The proof of the claim is contained in [21] for the case $\alpha = \frac{1}{n+2}$, see also [22] for the case $\alpha = 0$. The idea is similar to the proof of Lemma 3.2. But instead of considering the second boundary value problem with inhomogeneous term (3.14), we consider a weighted one

$$(4.16) \quad f_{m,j} = \begin{cases} f + \beta_m \chi_{D_m} & \text{in } \Omega' \\ H'_j(u - u_0) & \text{in } B_R \setminus \Omega' \end{cases}$$

where $H_j(t) = H(4^j t)$ given by (3.10), B_R is a large ball enclosing Ω' . By Lemma 3.3, there is a solution $u_{m,j}$ satisfying

$$(4.17) \quad |u_{m,j} - u_0| \leq 4^{-j}, \quad x \in B_R \setminus \Omega'.$$

By the convexity, $u_{m,j}$ sub-converges to a convex function u_m as $j \rightarrow \infty$ and $u_m = u_0$ in $B_R \setminus \Omega'$. Note that $u_m \in \mathcal{S}[u_0, \Omega']$ when restricted in Ω' , therefore, u_m converges to a convex function u_∞ in $\mathcal{S}[u_0, \Omega']$ as $m \rightarrow \infty$. Similarly, one can show that u_∞ is the maximizer of J_α over the set $\mathcal{S}[u_0, \Omega']$. By the uniqueness of maximizer, we have $u_\infty = u_0$ and obtain the claim. See [21, 22] for more details.

Now, suppose that ℓ is a line segment in \mathcal{G}_0 with both end points on $\partial\mathcal{G}_0$. By subtracting a linear function, we assume that $u_0 \geq 0$ and ℓ lies in $\{x_3 = 0\}$. From the definition of Ω' , we also have $\ell \subset \Omega'$ with both end points on $\partial\Omega' \cap \partial\Omega$. By a traslation and a dilation of the coordiantes, we may assume furthermore that

$$(4.18) \quad \ell = \{(0, x_2, 0) : -1 \leq x_2 \leq 1\}$$

with the endpoints $(0, \pm 1) \in \partial\Omega' \cap \partial\Omega$.

Since φ is smooth, uniformly convex in a neighborhood of Ω and $u_0 = \varphi$ on $\partial\Omega$, it follows

$$(4.19) \quad u_0(x) = \varphi(x) \leq \frac{C}{2}|x_1|^2, \quad x \in \partial\Omega' \cap \partial\Omega.$$

By the convexity of u_0 ,

$$(4.20) \quad u_0(x) \leq \frac{C}{2}|x_1|^2, \quad x \in \Omega'.$$

Consider the Legendre transform u_0^* of u_0 in $\Omega^* = D\varphi(\Omega)$, given by

$$(4.21) \quad u_0^*(y) = \sup\{x \cdot y - u_0(x), \quad x \in \Omega\}, \quad y \in \Omega^*.$$

Since both endpoints $(0, \pm 1) \in \partial\Omega' \cap \partial\Omega$, by the uniform convexity of φ , $0 \notin D\varphi(\partial\Omega)$. Hence $0 \in \Omega^*$ is an interior point. By (4.19), (4.20) we have

$$(4.22) \quad u_0^*(0, y_2) \geq |y_2|,$$

$$(4.23) \quad u_0^*(y) \geq \frac{1}{2C}y_1^2.$$

Therefore, $\det D^2 u_0^*$ is not bounded from above near the origin by Lemma 4.1.

But on the other hand, by the a priori estimate in Lemma 2.3, $\det D^2 u_0^*$ must be bounded. Indeed, consider the Legendre transform u_m^* of u_m . By the approximations (4.14), (4.15), and (2.10), u_m^* satisfies the equation

$$(4.24) \quad U^{*ij} w_{ij}^* = -f_m(Du^*) \det D^2 u^* \quad \text{in } \Omega_{\varepsilon_m}^*,$$

where $f_m = f + \beta_m \chi_{D_m}$ and

$$\Omega_{\varepsilon_m}^* = \{y \in \Omega^* : \text{dist}(y, \partial\Omega^*) > \varepsilon_m\}$$

with $\varepsilon_m \rightarrow 0$ as $m \rightarrow \infty$. By the growth estimates (4.22) and (4.23), u_0^* is strictly convex at 0, the set $\{u_0^* < h\}$ is strictly contained in Ω^* provided $h > 0$ is small. Note that u_m^* converges to u_0^* . By Lemma 2.3 we have the estimate

$$\det D^2 u_m^* \leq C_1$$

near the origin in Ω^* . Note also that in Lemma 2.3, the constant C_1 depends on $\inf f$ but not on $\sup f$. In other words, the large constant β_m in (4.13) does not affect the bound C_1 . Therefore, sending $m \rightarrow \infty$, we obtained

$$\det D^2 u_0^* \leq C$$

near the origin. This is in contradiction with the assertion that $\det D^2 u_0^*$ is not bounded from above near the origin. \square

5. REGULARITY

We can now give the proof of Theorem 1.1, which is divided into two parts:

5.1. $C^{1,\alpha}$ regularity. Assume that ψ is convex and satisfies $\psi < \varphi$ on $\partial\Omega$. Let u be the maximizer of (2.4) and \mathcal{G}_u be the graph of u over Ω . From Section 4 we know \mathcal{G}_u is strictly convex. The $C^{1,\alpha}$ estimate for strictly convex solutions of Monge-Ampère equations was obtained by Caffarelli [2]. Here we adopt a similar argument from [19].

For an arbitrary point on \mathcal{G}_u , by choosing appropriate coordinates and a rotation in \mathbb{R}^{n+1} , we assume it is the origin and $\mathcal{G}_u \subset \{x_3 \geq 0\}$, and near the origin \mathcal{G}_u is the graph of a strictly convex function u .

Lemma 5.1. *There exist positive constants α, β , and C such that*

$$(5.1) \quad C^{-1}|x|^{1+\beta} \leq u(x) \leq C|x|^{1+\alpha} \quad \text{near the origin.}$$

Proof. Denote $S_h^0 = \{x \in \Omega : u(x) < h\}$. By the strict convexity, $S_h^0 \Subset \Omega$ when $h > 0$ is small. We point out that the proof of strict convexity in Section 4 implies that u is C^1 smooth. In fact, if u is not C^1 at some point, by a rotation of axes we assume $\mathcal{G}_u \subset \{x_3 \geq a|x_1|\}$ for some constant $a > 0$. Let L be the intersection of \mathcal{G}_u with $\{x_3 = 0\}$. L could be a single point or a segment on x_2 -axis. From the proof of Lemma 4.2, by a contradiction argument, we can rule out the possibility of both cases, which implies that \mathcal{G}_u is C^1 smooth. Hence we have

$$(5.2) \quad \text{dist}\left(S_{h/2}^0, \partial S_h^0\right) \geq C_1,$$

or equivalently,

$$(5.3) \quad u(\theta x) \geq \frac{1}{2}u(x)$$

for any $x \in \partial S_h^0$, where $\theta = 1 - \frac{1}{2}C_1$. As h is any small constant, it follows that for any x near the origin,

$$(5.4) \quad u(x) \geq 2^{-k}u(\theta^{-k}x)$$

provided $\theta^{-k}x \in \Omega$. Hence we obtain the first inequality in (5.1) with β given by $\theta^{1+\beta} = 1/2$.

To prove the second inequality, we claim that there exists a constant $\sigma > 0$ such that for any small $h > 0$ and any $x \in \partial S_h^0$,

$$(5.5) \quad u\left(\frac{1}{2}x\right) < \frac{1-\sigma}{2}u(x).$$

Define α by $1 - \sigma = 2^{-\alpha}$. Then for any $x \in \partial\Omega$ and any $t \in (\frac{1}{2^{k+1}}, \frac{1}{2^k})$,

$$(5.6) \quad \begin{aligned} u(tx) &\leq 2^{-k}(1-\sigma)^k u(x) \\ &= (2^{-k})^{1+\alpha} u(x) \\ &\leq 2t^{1+\alpha} u(x). \end{aligned}$$

Hence $u \in C^{1,\alpha}$.

Inequality (5.5) follows from (5.3) as proved in [19]. For the reader's convenience, we include it here. Consider the convex function $g(t) = u(tx)$, $t \in [-1, 1]$. Replacing g by $g/g(1)$, we may

assume that $g(1) = 1$. Let $\psi(t) = g(t + \frac{1}{2}) - g'(\frac{1}{2})t - g(\frac{1}{2})$. Then $\psi(0) = 0, \psi \geq 0$. If $g(\frac{1}{2}) > \frac{1-\varepsilon}{2}$, by convexity we have $1 + \varepsilon \geq g'(\frac{1}{2}) \geq 1 - \varepsilon$ and $\psi(-\frac{1}{2}) \leq \varepsilon$. Applying (5.3) to ψ , we have $\psi(-\frac{1}{2}\theta^{-1}) \leq 2\psi(-\frac{1}{2}) \leq 2\varepsilon$. Hence $g(-\frac{1}{2}\theta^{-1} + \frac{1}{2}) < 0$ when $\varepsilon < \frac{1-\theta}{5}$, we reach a contradiction as $u \geq 0$. \square

We remark that the estimate (5.1) was also obtained in [14] for strictly c -convex solutions of general Monge-Ampère equations arising in the optimal transportation by a duality argument.

5.2. $C^{1,1}$ regularity. Assume that ψ is uniformly convex. Denote $T = \{x \in \Omega : u(x) = \psi(x)\}$ and $F = \Omega - T$. Let $\mathcal{G}_T, \mathcal{G}_F$ be the graph of u over T, F , respectively. For any point $p \in \partial\mathcal{G}_F$, we may choose a proper coordinate system such that p is the origin; and by a rotation in \mathbb{R}^{n+1} , we may also assume that $\{x_3 = 0\}$ is a tangent plane of \mathcal{G}_ψ . Therefore, $\psi(0) = 0, D\psi(0) = 0$, $u \geq \psi$ and ψ is uniformly convex.

Lemma 5.2. *Assume that ψ is uniformly convex. There exist two positive constants $C_1, C_2 > 0$ such that*

$$(5.7) \quad C_1|x|^2 \leq u(x) \leq C_2|x|^2.$$

Proof. The first inequality follows from the uniform convexity of ψ . That is

$$u(x) \geq \psi(x) \geq C_1|x|^2$$

as $\{x_3 = 0\}$ is the tangent plane of \mathcal{G}_ψ at the origin.

For the second inequality, suppose by contradiction that it is not true, then there is a sequence of points x_k with $|x_k| \rightarrow 0$ such that $u(x_k) \geq 2^k|x_k|^2$. We claim that

$$(5.8) \quad |N_u(E_{\varepsilon_k})| \geq C2^{k/2}\varepsilon_k^{n/2}$$

where $\varepsilon_k = u(x_k), E_\varepsilon = \{x \in \Omega : u(x) < \varepsilon\}$. To prove (5.8), by a rescaling

$$u \rightarrow \varepsilon_k^{-1}u, \quad \text{and } x \rightarrow \varepsilon_k^{-1/2}x,$$

we may assume $\varepsilon = 1$. Let v be a convex function defined on the entire \mathbb{R}^2 such that $v(0) = 0, v = u = 1$ on $\partial E_1 = \partial\{u < 1\}$, and v is homogeneous of degree 1. Then the graph of v is a convex cone with vertex at the origin. By the convexity of u we have

$$N_v(E_1) \subset N_u(E_1).$$

By the first inequality (5.7), we have

$$N_v(E_1) \supset B_{C_1^{1/2}}(0),$$

the ball of radius $C_1^{1/2}$. By the assumption that $1 = v(x_k) = u(x_k) > 2^k|x_k|^2$, the slope of v at x_k is greater than $2^{k/2}$. Hence there exists a point $\hat{p} \in N_v(E_1)$ such that $|\hat{p}| \geq 2^{k/2}$. Finally noting that $N_v(E_1) = N_v(\mathbb{R}^2)$ is a convex set as v is a convex cone, we obtain

$$|N_v(E_1)| \geq CC_1^{(n-1)/2}|\hat{p}| \geq C2^{k/2}.$$

By rescaling back, we then obtain $|N_u(E_{\varepsilon_k})| \geq C2^{k/2}\varepsilon_k^{n/2}$.

On the other hand, by the first inequality in (5.7) we have $|E_\varepsilon| \leq C\varepsilon^{n/2}$. Hence by the determinant estimate in §2.5 we have

$$|N_u(E_{\varepsilon_k})| = \int_{E_{\varepsilon_k}} \det D^2 u \leq C\varepsilon_k^{n/2}.$$

When k is sufficiently large, we reach a contradiction. \square

Corollary 5.1. *There is no line segment on \mathcal{G}_F with an endpoint on $\partial\mathcal{G}_F$.*

Now we prove the second part of Theorem 1.1.

Theorem 5.1. *Suppose that ψ is uniformly convex. Then u is $C^{1,1}$ smooth in a neighborhood of ∂F .*

Proof. When $\alpha = \frac{1}{n+2}$, the $C^{1,1}$ regularity was obtained in [16] for enclosed convex hypersurfaces with maximal affine area, where the affine invariant property plays a crucial role. But for general $0 \leq \alpha \leq \frac{1}{n+2}$, we need to rotate the graph \mathcal{G} in \mathbb{R}^{n+1} and use the a priori determinant estimates in Section 2. Note that the dimension two is needed in the proof of strict convexity, see Lemmas 4.2 and 4.4.

Let $p = (p_1, p_2, p_3)$ be a point on \mathcal{G}_F , close to $\partial\mathcal{G}_F$. Let $\delta = \text{dist}(p, \partial\mathcal{G}_F)$ (Euclidean distance). Choosing a proper coordinate system we suppose the origin is a point on $\partial\mathcal{G}_F$ and $|p| = \delta$. By a rotation transform, suppose furthermore that $\mathcal{G}_\psi \subset \{x_3 \geq 0\}$, and near the origin u satisfies (5.7).

Let $u_\delta(x) = \delta^{-2}u(\delta x)$ and let $p_\delta = (\frac{p_1}{\delta}, \frac{p_2}{\delta}, \frac{p_3}{\delta^2})$. Then by (5.7),

$$(5.9) \quad C_1|x|^2 \leq u_\delta(x) \leq C_2|x|^2.$$

From Section 4, u_δ is strictly convex near p_δ . By the a priori estimates in Section 2 and the approximation in Section 3, we then infer that there exist constants $C_1, C_2 > 0$ such that

$$C_1 I \leq D^2 u_\delta(\bar{p}) \leq C_2 I$$

for any \bar{p} near p_δ , where I is the unit matrix. The constants C_1 and C_2 are independent of δ . By our rescaling, $D^2 u(p) = D^2 u_\delta(p_\delta)$. Hence the second derivatives of u are uniformly bounded near ∂F . This complete the proof. \square

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